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SINGULAR SURFACES IN A LINEAR THERMO-ELASTIC DIELECTRIC MATERIAL

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Abstract—Singular surfaces in a linear thermo-elastic dielectric material are considered, where the constitutive equations of the elastic dielectric proposed by Toupin and the heat equations with finite wave velocities are combined. There exist six types of singular surfaces including a stationary one. The velocities, the coupled fields and the variation of the amplitudes of the surfaces with respect to time are investigated. It is found that the amplitude of the mechanical transverse wave rotates during propagation and at the stationary surface the amplitude of the electric field periodically reverses in direction and the one of the polarization field rotates elliptically with the same period.

1. INTRODUCTION

Since Toupin[1] proposed the theory of elastic dielectric, several studies have been made for the wave propagations in the materials. McCarthy[2] and McCarthy and Green[3] considered the acceleration waves in a hyperelastic dielectric material. Tokuoka and Kobayashi[4] considered the harmonic waves in a linear isotropic dielectric material. Recently Saito and Tokuoka[5], taking into account the relaxation of heat conduction, proposed the constitutive equations of a linear thermo-elastic dielectric material and investigate the harmonic waves with attenuation constants.

In this paper the singular surfaces in a slightly more general thermo-elastic dielectric material are investigated, where the equation of molecular equilibrium contains the term of gyration vector. Two remarkable phenomena can be observed even if the effect of heat conduction is neglected, that is, the rotation of the amplitude of the mechanical transverse wave occurs and a stationary surface of electric and polarization fields exists. These phenomena have not been reported until now, because for the occurrence of them it is necessary to take into consideration the singular surfaces and the generalized equation of molecular equilibrium.

In Section 2 the basic equations and the definition of the singular surface are given. In Section 3 the existance of the six types of singular surfaces are shown: a mechanical transverse wave, two thermo-mechanical longitudinal waves, a electro-magneto-polarization wave, a electro-magnetic wave and a stationary surface of electric and polarization fields. The velocities and the relations among the coupled fields of the singular surfaces are also discussed here. In Section 4 the variation of the amplitudes of the singular surfaces with respect to time are analyzed.

2. BASIC EQUATIONS AND DEFINITION OF SINGULAR SURFACE

We consider the following basic equations of a homogeneous isotropic linear thermo-elastic dielectric material:

$$\operatorname{rot}\mathbf{e}+\dot{\mathbf{b}}=\mathbf{0},\qquad \operatorname{div}\mathbf{b}=0, \tag{2.1}$$

$$\frac{1}{\mu_0}\operatorname{rot}\mathbf{b}-\epsilon_0\dot{\mathbf{e}}-\dot{\mathbf{p}}=\mathbf{0},\qquad \operatorname{div}(\epsilon_0\mathbf{e}+\mathbf{p})=0, \tag{2.2}$$

$$\mathbf{e}+(\dot{\mathbf{u}}\times\mathbf{B}_0)-\frac{1}{\epsilon_0\chi}\mathbf{p}+(\gamma\mathbf{B}_0\times\dot{\mathbf{p}})=\mathbf{0}, \tag{2.3}$$

$$\rho\ddot{\mathbf{u}}-(\lambda+\mu)\operatorname{grad}\operatorname{div}\mathbf{u}-\mu\nabla^2\mathbf{u}-(\dot{\mathbf{p}}\times\mathbf{B}_0)+\alpha_0\operatorname{grad}T=\mathbf{0}, \tag{2.4}$$

$$\tau\dot{\mathbf{q}}+\mathbf{q}+K\operatorname{grad}T=\mathbf{0}, \tag{2.5}$$

$$\alpha_0T_0\operatorname{tr}\dot{\boldsymbol{\epsilon}}+\beta_0T_0\dot{T}+\operatorname{div}\mathbf{q}=0, \tag{2.6}$$

where

$$\epsilon_{ij} \equiv \tfrac{1}{2}(u_{i,j} + u_{j,i}), \tag{2.7}$$

$$\alpha_0 \equiv \rho(3\lambda + 2\mu)\alpha, \tag{2.8}$$

$$\beta_0 \equiv \rho C_V / T_0, \tag{2.9}$$

and we assume that the external forces and the heat supplies do not exist. In the above equations the dependent variables of co-ordinates and time are the magnetic field $\mathbf{b}$, the electric field $\mathbf{e}$, the polarization field $\mathbf{p}$, the displacement $\mathbf{u}$, the heat flux $\mathbf{q}$ and the temperature T, and they denote the small derivations from an equilibrium static state with a constant magnetic field. The known constants are the magnetic permeability of vacuum μ_0, the dielectric constant of vacuum ϵ_0, the external magnetic field $\mathbf{B}_0$, the coefficient of the gyration vector expressed in terms of $\mathbf{B}_0$ γ, the polarizability χ, the density ρ, the elastic constants λ and μ, the coefficient of thermal expansion α, the relaxation time of heat conduction τ, the specific heat at constant volume C_V, the temperature of an equilibrium state T_0, and the heat conductivity K.

Henceforth we shall be concerned with the plane singular surfaces having the following properties:

(i) All the dependent variables and the first derivatives of $\mathbf{u}$ are continuous everywhere.

(ii) The first derivatives of the dependent variables except $\mathbf{u}$ and the second derivatives of $\mathbf{u}$ have finite jump discontinuities at the surfaces but they are continuous everywhere else.

Then the first and the second compatibility conditions for a continuous function f of co-ordinates and time at a singular surface are

$$[f_{,i}] = \bar{f} n_i, \qquad [\dot{f}] = -U\bar{f}, \tag{2.10}$$

$$[f_{,ij}] = \bar{\bar{f}} n_i n_j, \qquad [\dot{f}_{,i}] = (\dot{\bar{f}} - U\bar{\bar{f}}) n_i, \qquad [\ddot{f}] = U^2 \bar{\bar{f}} - 2U\dot{\bar{f}}, \tag{2.11}$$

where

$$\bar{f} \equiv [f_{,i}] n_i, \qquad \bar{\bar{f}} \equiv [f_{,ij}] n_i n_j. \tag{2.12}$$

Here a bracket denotes the jump of the quantity within it at the surface and U and $\mathbf{n}$ are, respectively, the normal velocity and the unit normal vector, which is determined as in the propagation direction in the case $U \neq 0$. The quantity $\bar{f}$ is called the amplitude of f at the surface. In the following sections, when we wish to distinguish between singular surfaces with the velocities $U \neq 0$ and $U = 0$, we use the terms of a wave and a stationary surface, respectively.

3. VELOCITIES AND COUPLED FIELDS OF SINGULAR SURFACES

Taking the jumps of eqns (2.1)–(2.6) by means of eqns (2.10) and (2.11) yields

$$\bar{\mathbf{e}} \times \mathbf{n} + U\bar{\mathbf{b}} = \mathbf{0}, \qquad \bar{\mathbf{b}} \cdot \mathbf{n} = 0, \tag{3.1}$$

$$\frac{1}{\mu_0} \bar{\mathbf{b}} \times \mathbf{n} - \epsilon_0 U \bar{\mathbf{e}} - U\bar{\mathbf{p}} = \mathbf{0}, \qquad \bar{\mathbf{e}} \cdot \mathbf{n} + \bar{\mathbf{p}} \cdot \mathbf{n} = 0, \tag{3.2}$$

$$\gamma U \mathbf{B}_0 \times \bar{\mathbf{p}} = \mathbf{0}, \tag{3.3}$$

$$(\rho U^2 - \mu)\bar{\mathbf{v}} - (\lambda + \mu)(\bar{\mathbf{v}} \cdot \mathbf{n})\mathbf{n} - \alpha_0 U \bar{T} \mathbf{n} = \mathbf{0}, \tag{3.4}$$

$$\tau U \bar{\mathbf{q}} - K \bar{T} \mathbf{n} = \mathbf{0}, \tag{3.5}$$

$$\alpha_0 T_0 \bar{\mathbf{v}} \cdot \mathbf{n} - \beta_0 T_0 U \bar{T} + \bar{\mathbf{q}} \cdot \mathbf{n} = 0, \tag{3.6}$$

where

$$\mathbf{v} \equiv \dot{\mathbf{u}}. \tag{3.7}$$

While taking the jumps of the co-ordinate derivatives of the inner product of eqn (2.3) by $\mathbf{B}_0$ yields another equation:

$$\bar{\mathbf{e}} \cdot \mathbf{B} - \frac{1}{\epsilon_0 \chi} \bar{\mathbf{p}} \cdot \mathbf{B}_0 = 0. \tag{3.8}$$

If the singular surface is a wave, eqns (3.3) and (3.8) can be combined into the equivalent equation:

$$\epsilon_0\chi(\bar{\mathbf{e}}\cdot\mathbf{B}_0)\mathbf{B}_0-(\mathbf{B}_0\cdot\mathbf{B}_0)\bar{\mathbf{p}}=\mathbf{0}. \tag{3.9}$$

Then the above jump equations are expressed as

$$R_{\alpha\beta}\bar{a}_\beta=0, \qquad (\alpha,\beta=1,2,\ldots,16) \tag{3.10}$$

where

$$\mathbf{a}\equiv(\mathbf{e},\mathbf{b},\mathbf{p},\mathbf{v},\mathbf{q},T), \tag{3.11}$$

$$\|R_{\alpha\beta}\|\equiv\begin{Vmatrix}\mathbf{A} & \mathbf{0}\\ \mathbf{0} & \mathbf{B}\end{Vmatrix}\equiv\begin{Vmatrix}-\mathbf{n}\times & U\mathbf{1} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0}\\ \epsilon_0\mu_0U\mathbf{1} & \mathbf{n}\times & \mu_0U\mathbf{1} & \mathbf{0} & \mathbf{0} & \mathbf{0}\\ \epsilon_0\chi\mathbf{B}_0\otimes\mathbf{B}_0 & \mathbf{0} & -(\mathbf{B}_0\cdot\mathbf{B}_0)\mathbf{1} & \mathbf{0} & \mathbf{0} & \mathbf{0}\\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \begin{matrix}(\rho U^2-\mu)\mathbf{1}\\-(\lambda+\mu)\mathbf{n}\otimes\mathbf{n}\end{matrix} & \mathbf{0} & -\alpha_0U\mathbf{n}\\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \tau U\mathbf{1} & -K\mathbf{n}\\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \alpha_0T_0\mathbf{n} & \mathbf{n} & -\beta_0T_0U\end{Vmatrix} \tag{3.12}$$

and $\mathbf{n}\times$ denotes the 3×3 matrix which corresponds to the vector product by $\mathbf{n}$. It is well known that the determinant of the coefficients matrix of eqn (3.10) should vanish for any wave, or equivalently

$$\det\mathbf{A}=0 \qquad \text{or} \qquad \det\mathbf{B}=0 \tag{3.13}$$

should holds. The velocity and the coupled fields of the wave are given, respectively, by U satisfying eqn (3.13) and by the non-zero amplitudes satisfying eqn (3.10) for the velocity. Since we may assume that in general eqns $(3.13)_{1,2}$ do not hold simultaneously for a single value of U, we have

$$\bar{\mathbf{v}},\bar{\mathbf{q}}=\mathbf{0}, \qquad \bar{T}=0 \qquad \text{or} \qquad \bar{\mathbf{e}}=\bar{\mathbf{b}}=\bar{\mathbf{p}}=\mathbf{0}, \tag{3.14}$$

respectively, for the values of U satisfying eqns $(3.13)_{1,2}$. Thus we can conclude that there exist no couplings between the thermo-mechanical fields and the electro–magneto-polarization fields for any wave.

The matrix $\mathbf{B}$ in eqn $(3.13)_2$ is found to coincide with the one for the linear thermo–elastic material considered by Tokuoka[6] except the definitions of the known constants, so that by applying the obtained result the following three types of waves exist for eqn $(3.13)_2$.

Mechanical transverse wave

If we suppose that

$$\bar{\mathbf{v}}\times\mathbf{n}\neq\mathbf{0}, \tag{3.15}$$

eqn (3.4) imposes

$$U^2=\frac{\mu}{\rho}. \tag{3.16}$$

The velocity is equal to the one of the usual transverse wave. For this wave we obtain from eqns (3.4)–(3.6)

$$\bar{\mathbf{v}}\cdot\mathbf{n}=\bar{T}=0, \qquad \bar{\mathbf{q}}=\mathbf{0}. \tag{3.17}$$

Thermo–mechanical longitudinal waves

If we suppose that

$$\bar{\mathbf{v}}\cdot\mathbf{n}, \qquad \bar{\mathbf{q}}\cdot\mathbf{n}, \qquad \bar{T}\neq0, \tag{3.18}$$

then we have from eqns (3.4)–(3.6)

$$U^2 = \tfrac{1}{2}\{c_H^2 + c_L^2 + \gamma_0 \pm [(c_H^2 + c_L^2 + \gamma_0)^2 - 4c_H^2 c_L^2]^{1/2}\}, \tag{3.19}$$

where

$$c_H^2 \equiv \frac{K}{\tau B_0 T_0}, \qquad c_L^2 \equiv \frac{\lambda + 2\mu}{\rho}, \qquad \gamma_0 \equiv \frac{\alpha_0^2}{\rho \beta_0}. \tag{3.20}$$

For these waves eqns (3.4) and (3.5) impose

$$\bar{\mathbf{v}} \times \mathbf{n} = \bar{\mathbf{q}} \times \mathbf{n} = \mathbf{0}. \tag{3.21}$$

As was mentioned above, the amplitudes of the electro–magneto-polarization fields vanish at these three waves, and therefore the velocities and the coupled fields of them coincide completely with those of the linear thermo-elastic material. It will be however shown in the next section that the variation of the mechanical transverse wave with respect to time differs from that of the thermo-elastic material. We shall next consider the case of eqn $(3.13)_1$.

Electro–magneto-polarization wave

Suppose that the amplitude of the polarization field does not vanish at a wave, then it should be parallel to the external magnetic field from eqn (3.3). The following equations are derived from eqns (3.1) and (3.2) for any singular surface:

$$(1 - \epsilon_0\mu_0 U^2)\bar{\mathbf{b}} = \mu_0 U \mathbf{n} \times \bar{\mathbf{p}}, \tag{3.22}$$

$$\epsilon_0(1 - \epsilon_0\mu_0 U^2)\bar{\mathbf{e}} = \epsilon_0\mu_0 U^2 \bar{\mathbf{p}} - (\bar{\mathbf{p}} \cdot \mathbf{n})\mathbf{n}. \tag{3.23}$$

Eliminating $\bar{\mathbf{e}}$ from eqns (3.9) and (3.23), and taking the inner product of the result by $\mathbf{B}_0$ lead to the equation:

$$[(1 + \chi)\epsilon_0\mu_0 U^2 - (1 + \chi \cos^2\theta)](\bar{\mathbf{p}} \cdot \mathbf{B}_0) = 0. \tag{3.24}$$

Here and henceforth θ denotes the angle between $\mathbf{n}$ and $\mathbf{B}_0$. Since the inner product of $\bar{\mathbf{p}}$ and $\mathbf{B}_0$ does not vanish for the wave where $\bar{\mathbf{p}}$ is parallel to $\mathbf{B}_0$, eqn (3.24) gives the velocity of the wave:

$$U^2 = \frac{1 + \chi\cos^2\theta}{1 + \chi} c^2, \tag{3.25}$$

where $c \equiv (\epsilon_0\mu_0)^{-1/2}$ denotes the velocity of light in vacuum. Figure 1 shows that the velocity is always smaller than or equal to that of light and that the retardation increases as χ increases or as θ draws to $\pi/2$.

The amplitude of the electric field lies on the plane spanned by $\mathbf{n}$ and $\bar{\mathbf{p}}$ in view of eqn (3.23). Let ϕ be the angle between $\mathbf{n}$ and $\bar{\mathbf{e}}$ measured in the same direction as θ, see Fig. 2. To determine the angle ϕ, we rewrite eqns $(3.2)_2$ and (3.8) as

$$\epsilon_0 \cos\phi\, \bar{e} \pm \cos\theta \bar{p} = 0, \tag{3.26}$$

$$\epsilon_0\chi \cos(\phi - \theta)\, \bar{e} \mp \bar{p} = 0. \tag{3.27}$$

The assumption that $\bar{\mathbf{p}}$ does not vanish for this wave imposes that the determinant of the coefficients matrix of $\bar{e}$ and $\bar{p}$ must be zero. Hence we finally obtain

$$\phi = \arctan\left(-\frac{1 + \chi\cos^2\theta}{\chi \sin\theta\cos\theta}\right). \tag{3.28}$$

Figure 3 shows that $\bar{\mathbf{e}}$ is nearly perpendicular to $\mathbf{n}$ for small χ and that it is nearly perpendicular

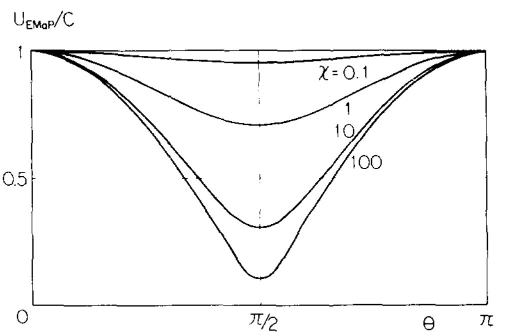

Fig. 1.

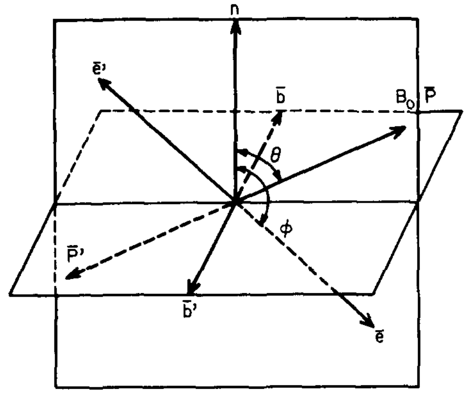

Fig. 2.

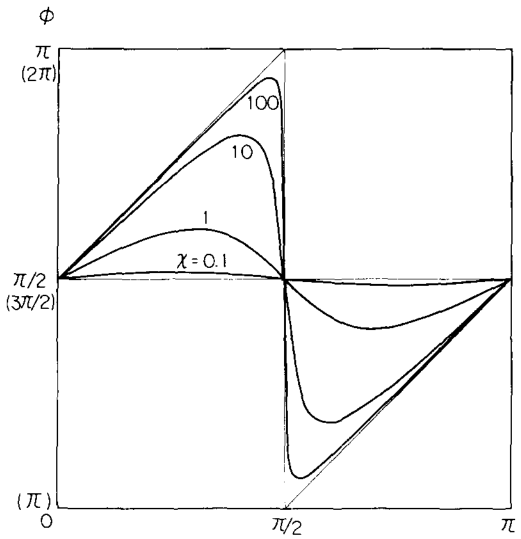

Fig. 3.

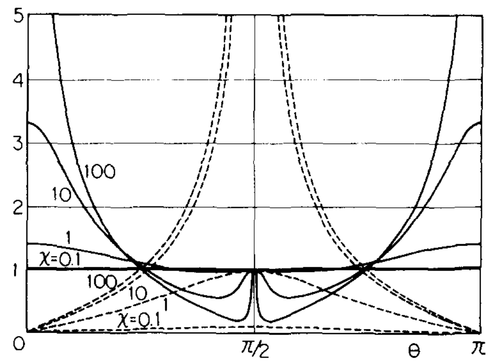

Fig. 4.

Fig. 1. Variation of the velocity of the electro-magneto-polarization wave.

Fig. 2. Coupled fields of the electro-magneto-polarization wave.

Fig. 3. Variation of the angle ϕ between $\bar{\mathbf{e}}$ and $\mathbf{n}$ at the electro-magneto-polarization wave when the directions of $\bar{\mathbf{p}}$ and $\mathbf{B}_0$ are the same. The values of ϕ in the parentheses correspond to the case where the directions of $\bar{\mathbf{p}}$ and $\mathbf{B}_0$ are the opposite.

Fig. 4. Variation of the ratios among $\bar{e}$, $\bar{b}$ and $\bar{p}$ at the electro-magneto-polarization wave, where solid and broken lines denote, respectively, $c\bar{b}/\bar{e}$ and $\bar{p}/\epsilon_0\bar{e}$.

to $\mathbf{B}_0$ for large χ and except for θ near $\pi/2$. It is also found that for large χ the direction of $\bar{\mathbf{e}}$ is very sesitive to the small change of θ near $\pi/2$.

In the meanwhile the amplitude of the magnetic field is perpendicular to the plane spanned by $\mathbf{n}$ and $\bar{\mathbf{e}}$ from eqn $(3.1)_1$, see Fig. 2. Hence we can conclude that all the coupled fields of this wave, when $\mathbf{B}_0$ is not parallel to $\mathbf{n}$, are linearly polarized in the directions determined by $\mathbf{n}$, θ and χ. The variation of the ratios among $\bar{e}$, $\bar{b}$ and $\bar{p}$ with respect to χ and θ shown in Fig. 4 are given by eqns (3.1), (3.25) and (3.26). The ratios $\bar{b}/\bar{e}$ and $\bar{p}/\bar{e}$ take their maximum values, respectively, when $\mathbf{B}_0$ is parallel and perpendicular to $\mathbf{n}$. When $\mathbf{B}_0$ is parallel to $\mathbf{n}$, $\bar{\mathbf{p}}$ vanishes and the ratio $\bar{b}/\bar{e}$ coincides with that of the usual electro-magnetic wave. In view of Fig. 1, the velocity is also found to coincide with that of the electro-magnetic wave in this case. Hence we can state that the electro-magneto-polarization wave reduces to the electro-magnetic wave when $\mathbf{B}_0$ is parallel to $\mathbf{n}$.

Electro-magnetic wave

If $\mathbf{B}_0$ is not parallel to $\mathbf{n}$ and if a wave has a different velocity from the ones obtained up to now, the above discussion imposes that

$$\bar{\mathbf{v}}, \bar{\mathbf{q}}, \bar{\mathbf{p}} = \mathbf{0}, \qquad \bar{T} = 0 \tag{3.29}$$

for this wave. Supposing that

$$\bar{\mathbf{e}}, \bar{\mathbf{b}} \neq \mathbf{0} \tag{3.30}$$

yields

$$U^2 = c^2 \tag{3.31}$$

from eqns (3.22) and (3.23). Thus this wave is a electro-magnetic wave with the velocity of light. From eqns $(3.2)_2$, (3.8) and (3.29), we have

$$\bar{\mathbf{e}} \cdot \mathbf{n} = 0, \qquad \bar{\mathbf{e}} \cdot \mathbf{B}_0 = 0, \tag{3.32}$$

which implies that $\bar{\mathbf{e}}$ is linearly polarized in the directions of $\pm\mathbf{n} \times \mathbf{B}_0$. While $\bar{\mathbf{b}}$ is also linearly polarized in the directions of $\pm\mathbf{n} \times (\mathbf{n} \times \mathbf{B}_0)$, see Fig. 5.

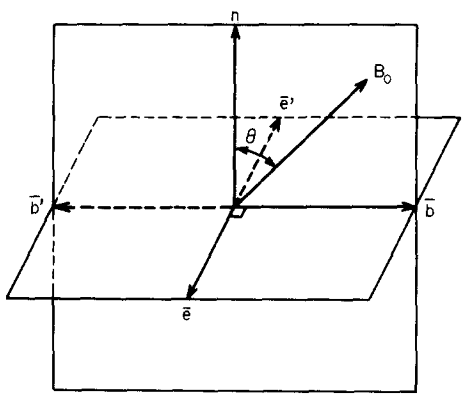

Fig. 5. Coupled fields of the electro-magnetic wave.

When $\mathbf{B}_0$ is parallel to $\mathbf{n}$, we have already proved that the electro-magnetic wave with the velocity given by eqn (3.31) can exist. Then $\bar{\mathbf{e}}$ may point in any direction being perpendicular to $\mathbf{n}$, and eqn (3.1) implies that $\bar{\mathbf{b}}$ is perpendicular to $\mathbf{n}$ and $\bar{\mathbf{e}}$.

Stationary surface of electric and polarization fields

Up to now we have presented all the waves which can exist, and so if another singular surface may exist, it must be a stationary one. To investigate it, let us put $U = 0$ in the basic jump eqns (3.1)–(3.6) and (3.8), which are valid for any singular surface. Then the coefficients matrix for $\bar{\mathbf{v}}$, $\bar{\mathbf{q}}$ and $\bar{T}$ is again given by $\mathbf{B}$ in eqn (3.12). Since $U = 0$ does not satisfy $\det \mathbf{B} = 0$ from the discussions for the mechanical transverse and the thermo-mechanical longitudinal waves, we have

$$\bar{\mathbf{v}}, \bar{\mathbf{q}} = \mathbf{0}, \qquad \bar{T} = 0 \tag{3.33}$$

at this surface. While the coefficients matrix for $\bar{\mathbf{e}}$, $\bar{\mathbf{b}}$ and $\bar{\mathbf{p}}$ differs from $\mathbf{A}$ in eqn (3.12), because eqn (3.9) which is involved in eqn (3.10) is not valid for the stationary surface. Putting $U = 0$ in eqns (3.1) and (3.2) yields

$$\bar{\mathbf{b}} = \mathbf{0}, \qquad \bar{\mathbf{e}} \times \mathbf{n} = \mathbf{0}, \tag{3.34}$$

where the latter equation states that $\bar{\mathbf{e}}$ is linearly polarized in the directions of $\pm\mathbf{n}$. Equation (3.8) is reduced to, by means of eqn $(3.34)_2$,

$$\epsilon_0 \chi \cos\theta\,(\bar{\mathbf{e}} \cdot \mathbf{n}) - \bar{\mathbf{p}} \cdot \mathbf{m} = 0, \tag{3.35}$$

where $\mathbf{m}$ denotes the unit vector in the direction of $\mathbf{B}_0$. Eliminating $\bar{\mathbf{e}}$ from eqns $(3.2)_2$ and (3.35) yields

$$\bar{\mathbf{p}} \cdot (\chi \cos \theta \mathbf{n} + \mathbf{m}) = 0, \tag{3.36}$$

which means that $\bar{\mathbf{p}}$ is perpendicular to the vector $\mathbf{m} + \chi \cos \theta \mathbf{n}$. Let ψ be the angle between this vector and $\mathbf{n}$ measured in the same direction as θ. It is easy to obtain

$$\psi = \arctan (\tan \theta / 1 + \chi), \tag{3.37}$$

see Fig. 6. In view of Fig. 7, we are found that $\bar{\mathbf{p}}$ is nearly perpendicular to $\mathbf{B}_0$ for small χ and that it is nearly perpendicular to $\mathbf{n}$ for large χ and except for θ near $\pi/2$.

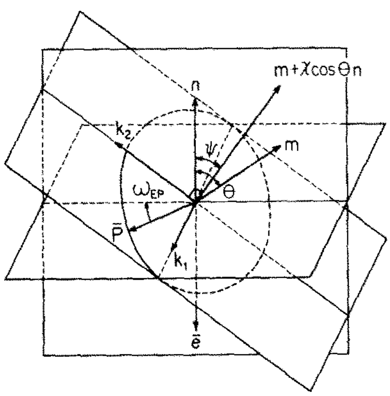

Fig. 6.

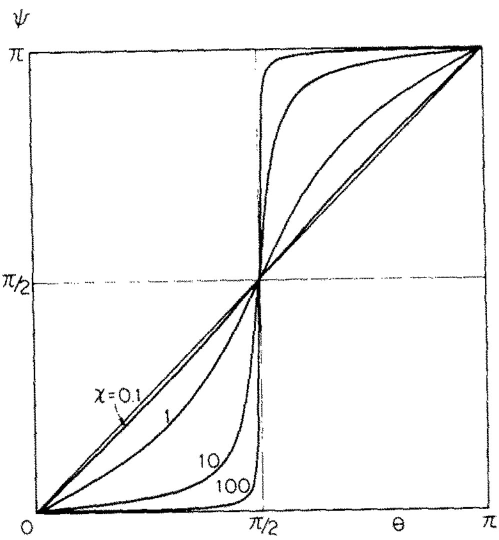

Fig. 7.

Fig. 6. Coupled fields of the stationary surface, where $\bar{\mathbf{p}}$ and $\bar{\mathbf{e}}$ do not lie in the same side of the plane being perpendicular to $\mathbf{n}$.

Fig. 7. Variation of the angle ψ between $\mathbf{n}$ and $\mathbf{m} + \chi \cos \theta \mathbf{n}$ at the stationary surface.

4. VARIATION OF AMPLITUDES OF SINGULAR SURFACES

To obtain the differential equations which govern the variation of the amplitudes, we take the jumps of the first time derivatives of eqns (2.1)–(2.6) and the second time derivative of the inner product of eqn (2.3) by $\mathbf{B}_0$. The obtained equations can be expressed in the following form for the waves:

$$P_{\alpha\beta}\dot{\bar{a}}_\beta = Q_{\alpha\beta}\bar{a}_\beta + UR_{\alpha\beta}\bar{\bar{a}}_\beta, \qquad (\alpha, \beta = 1, 2, \ldots, 16) \tag{4.1}$$

where

$$\|P_{\alpha\beta}\| = \left\| \begin{array}{cccccc} \mathbf{0} & U\mathbf{1} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \epsilon_0\mu_0 U\mathbf{1} & \mathbf{0} & \mu_0 U\mathbf{1} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \begin{array}{c}(\rho U^2 + \mu)\mathbf{1} \\ +(\lambda + \mu)\mathbf{n} \otimes \mathbf{n}\end{array} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \tau U\mathbf{1} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & -\beta_0 T_0 U \end{array} \right\| \tag{4.2}$$

$$\gamma\|Q_{\alpha\beta}\| = \left\| \begin{array}{cccccc} \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ -\mathbf{B}_0 \times & \mathbf{0} & \dfrac{1}{\epsilon_0\chi}\mathbf{B}_0 \times & (\mathbf{B}_0 \times)^2 & \mathbf{0} & \mathbf{0} \\ U^2\mathbf{1} & \mathbf{0} & \dfrac{-U^2}{\epsilon_0\chi}\mathbf{1} & -U^2\mathbf{B}_0 \times & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & -\gamma U\mathbf{1} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{array} \right\| \tag{4.3}$$

and $R_{\alpha\beta}$ is defined by eqn (3.12). In the above derivation we used the relation $R_{\alpha\beta}\dot{a}_\beta = 0$ obtained by differentiating eqn (3.10) with respect to time. Let $\mathbf{1}^\gamma$ be the left null vectors of $R_{\alpha\beta}$ corresponding to the wave, then the differential equations for the amplitudes are obtained by taking the inner products of eqn (4.1) by $\mathbf{1}^\gamma$:

$$(1_\alpha{}^\gamma P_{\alpha\beta})\dot{\bar{a}}_\beta = (1_\alpha{}^\gamma Q_{\alpha\beta})\bar{a}_\beta \qquad (\alpha, \beta = 1, 2, \ldots, 16). \tag{4.4}$$

In what follows we shall determine the vectors $\mathbf{1}^\gamma$ by the heuristic method.

Mechanical transverse wave

For this wave we take the vector product of the part $\alpha = 10, 11, 12$ of eqn (4.1) by $\mathbf{n}$, instead of the inner products by null vectors. That is, we have

$$(\rho U^2 + \mu)\dot{\bar{\mathbf{v}}} \times \mathbf{n} = -\frac{U^2}{\gamma}(\mathbf{B}_0 \times \bar{\mathbf{v}}) \times \mathbf{n}, \tag{4.5}$$

where we have used the relations $\bar{\mathbf{e}} = \bar{\mathbf{b}} = \bar{\mathbf{p}} = \mathbf{0}$. Equation (4.5) is reduced to, by eqns (3.16) and (3.17),

$$\dot{\bar{\mathbf{v}}} \times \mathbf{n} = -\omega_{Me}\bar{\mathbf{v}}, \tag{4.6}$$

where

$$\omega_{Me} \equiv \frac{\mathbf{B}_0 \cdot \mathbf{n}}{2\rho\gamma}. \tag{4.7}$$

Equation (4.6) is easily solved by use of the perpendicularity of $\bar{\mathbf{v}}$ to $\mathbf{n}$ under the condition $\bar{\mathbf{v}} = \bar{\mathbf{v}}_0$ at $t = 0$:

$$\bar{\mathbf{v}} = \cos \omega_{Me}t\bar{\mathbf{v}}_0 + \sin \omega_{Me}t\bar{\mathbf{v}}_0 \times \mathbf{n}. \tag{4.8}$$

Thus the amplitude does not vary its magnitude but rotates with the angular velocity ω_{Me} on the plane being perpendicular to $\mathbf{n}$. In view of eqn (4.7) we can state that the amplitude of this wave is right or left circularly polarized, respectively, when the angle θ between $\mathbf{B}_0$ and $\mathbf{n}$ is less or larger than $\pi/2$. When $\mathbf{B}_0$ is perpendicular to $\mathbf{n}$, the rotation of the amplitude does not occur. The absolute value of ω_{Me} takes the maximum value when $\mathbf{B}_0$ is parallel to $\mathbf{n}$.

Thermo-mechanical longitudinal waves

Taking the inner product of the part $\alpha = 10, 11, \ldots, 16$ of eqn (4.1) by the vector:

$$\mathbf{1}_{TMe} \equiv \left(\frac{\alpha_0 T_0}{\rho U^2 - 2\mu - \lambda}\mathbf{n}, \frac{1}{\tau U}\mathbf{n}, -1\right) \tag{4.9}$$

yields

$$\frac{\alpha_0 T_0(\rho U^2 + 2\mu + \lambda)}{\rho U^2 - 2\mu - \lambda}\dot{\bar{\mathbf{v}}} \cdot \mathbf{n} + \dot{\bar{\mathbf{q}}} \cdot \mathbf{n} + \beta_0 T_0 U\dot{\bar{T}} = -\frac{\alpha_0 T_0 U^2}{\gamma(\rho U^2 - 2\mu - \lambda)}(\mathbf{B}_0 \times \bar{\mathbf{v}}) \cdot \mathbf{n} - \frac{1}{\tau}\bar{\mathbf{q}} \cdot \mathbf{n}. \tag{4.10}$$

The first term in the right of eqn (4.10) vanishes from eqn (3.21), and by eliminating $\bar{\mathbf{q}}$ and $\bar{T}$ by means of eqns (3.5) and (3.6) from it we finally obtain

$$\dot{\bar{\mathbf{v}}} \cdot \mathbf{n} = -\delta_{TMe}\bar{\mathbf{v}} \cdot \mathbf{n} \tag{4.11}$$

where

$$\delta_{TMe} \equiv \frac{\gamma_0 c_H{}^2}{2\tau[(U^2 - c_H{}^2)^2 + \gamma_0 c_H{}^2]} \tag{4.12}$$

is positive and U is given by eqn (3.19). The solution of eqn (4.11) under the condition $\bar{\mathbf{v}} = \bar{\mathbf{v}}_0$ at

$t = 0$ is

$$\bar{\mathbf{v}} = \bar{\mathbf{v}}_0 \exp(-\delta_{TMe} t). \qquad (4.13)$$

It is easy to show that $\bar{\mathbf{q}}$ and $\bar{T}$ are also expressed in the same form as eqn (4.13) Thus the amplitudes of the coupled fields of these waves decay exponentially in time. Since the two damping constants precisely coincide with those of the thermo–elastic material considered in [6], we are found that the variation of the amplitudes of these waves are not affected by the electro–magnetic properties of the material as well as the velocities of them.

Electro–magneto-polarization wave

According to the discussion in the last section, we may assume that $\mathbf{B}_0$ is not parallel to $\mathbf{n}$ for this wave. Taking the inner product of the part $\alpha = 1, 2, \ldots, 9$ of eqn (4.1) by the vector:

$$\mathbf{1}_{EMaP} \equiv \left\{\mathbf{n} \times \mathbf{B}_0, U\left[\mathbf{B}_0 - \frac{c^2}{U^2}(\mathbf{B}_0 \cdot \mathbf{n})\mathbf{n}\right], \frac{\mu_0 U^2}{B_0^2}\left[\mathbf{B}_0 - \frac{c^2}{U^2}(\mathbf{B}_0 \cdot \mathbf{n})\mathbf{n}\right]\right\} \qquad (4.14)$$

leads to the equation:

$$U(\mathbf{n} \times \mathbf{B}_0) \cdot \dot{\bar{\mathbf{b}}} + \epsilon_0 \mu_0 U^2 \left[\mathbf{B}_0 - \frac{c^2}{U^2}(\mathbf{B}_0 \cdot \mathbf{n})\mathbf{n}\right] \cdot \dot{\bar{\mathbf{e}}} + \mu_0 U^2 \left[\mathbf{B}_0 - \frac{c^2}{U^2}(\mathbf{B}_0 \cdot \mathbf{n})\mathbf{n}\right] \cdot \dot{\bar{\mathbf{p}}}$$

$$= -\frac{\mu_0 U^2}{\gamma B_0^2}\left[\mathbf{B}_0 - \frac{c^2}{U^2}(\mathbf{B}_0 \cdot \mathbf{n})\mathbf{n}\right] \cdot (\mathbf{B}_0 \times \bar{\mathbf{e}}) + \frac{\mu_0 U^2}{\epsilon_0 \chi \gamma B_0^2}\left[\mathbf{B}_0 - \frac{c^2}{U^2}(\mathbf{B}_0 \cdot \mathbf{n})\mathbf{n}\right] \cdot (\mathbf{B}_0 \times \bar{\mathbf{p}}). \qquad (4.15)$$

In the last section we showed that $\mathbf{B}_0$, $\mathbf{n}$, $\bar{\mathbf{p}}$ and $\bar{\mathbf{e}}$ are always situated on a plane, so that the vectors $\mathbf{B}_0 \times \bar{\mathbf{e}}$ and $\mathbf{B}_0 \times \bar{\mathbf{p}}$ should be the zero vector or perpendicular to the four vectors, see again Fig. 2. Hence the right of eqn (4.15) vanishes. Eliminating $\bar{\mathbf{b}}$ and $\bar{\mathbf{p}}$ from it by means of eqn $(3.1)_1$ and (3.9), we obtain

$$\dot{\bar{\mathbf{e}}} \cdot [\mathbf{B}_0 - (\mathbf{B}_0 \cdot \mathbf{n})\mathbf{n}] = 0, \qquad (4.16)$$

or equivalently

$$\dot{\bar{e}} \sin\theta \sin\phi = 0. \qquad (4.17)$$

Since θ is not equal to 0 or π in this case and so is ϕ in view of Fig. 3, eqn (4.17) implies that the magnitude of $\bar{\mathbf{e}}$ remains constant in time. The direction of $\bar{\mathbf{e}}$ is also constant from eqn (3.28), and therefore $\bar{\mathbf{e}}$ is a constant vector. The other coupled fields $\bar{\mathbf{b}}$ and $\bar{\mathbf{p}}$ are also constant vectors from eqns $(3.1)_1$ and (3.9).

Electro–magnetic wave

We first consider the case where $\mathbf{B}_0$ is not parallel to $\mathbf{n}$. Taking the inner product of the part $\alpha = 1, 2, \ldots, 9$ of eqn (4.1) by the vector:

$$\mathbf{1}_{EMa} \equiv \left[\mathbf{B}_0 - (\mathbf{B}_0 \cdot \mathbf{n})\mathbf{n}, U(\mathbf{B}_0 \times \mathbf{n}), \frac{\mu_0 U^2}{B_0^2}(\mathbf{B}_0 \times \mathbf{n})\right] \qquad (4.18)$$

yields

$$U(\mathbf{B}_0 \cdot \dot{\bar{\mathbf{b}}}) - U(\mathbf{B}_0 \cdot \mathbf{n})(\dot{\bar{\mathbf{b}}} \cdot \mathbf{n}) + (\mathbf{B}_0 \times \mathbf{n}) \cdot \dot{\bar{\mathbf{e}}} = -\frac{\mu_0 U^2}{\gamma B_0^2}(\mathbf{B}_0 \times \mathbf{n}) \cdot (\mathbf{B}_0 \times \bar{\mathbf{e}}). \qquad (4.19)$$

Figure 5 implies that the vectors $\mathbf{B}_0 \times \mathbf{n}$ and $\mathbf{B}_0 \times \bar{\mathbf{e}}$ make the right angle and hence the right of eqn (4.19) vanishes. By eliminating $\bar{\mathbf{b}}$ by means of eqn $(3.1)_1$, eqn (4.19) is reduced to

$$(\mathbf{B}_0 \times \mathbf{n}) \cdot \dot{\bar{\mathbf{e}}} = 0. \qquad (4.20)$$

Since $\bar{\mathbf{e}}$ is parallel to $\mathbf{B}_0 \times \mathbf{n}$ from (3.32), eqn (4.20) implies that $\bar{\mathbf{e}}$ is a constant vector in time. The amplitude $\bar{\mathbf{b}}$ is also a constant vector in time from eqn $(3.1)_1$.

When $\mathbf{B}_0$ is parallel to $\mathbf{n}$, replacing $\mathbf{1}_{EMa}$ in the above process by the vector:

$$\mathbf{1}'_{EMa} \equiv \left(\mathbf{n} \times \mathbf{c},\ U\mathbf{c},\ \frac{\mu_0 U^2}{B_0^2}\mathbf{c}\right), \tag{4.21}$$

where $\mathbf{c}$ is an arbitrary vector not being parallel or perpendicular to $\mathbf{n}$, leads to the same result.

Stationary surface of electric and polarization fields

For this surface eqn (4.1) is not valid and the desired differential equation for the amplitudes is given by taking the jumps of the first co-ordinate derivatives of eqn (2.3):

$$\gamma \mathbf{B}_0 \times \dot{\bar{\mathbf{p}}} = \frac{1}{\epsilon_0 \chi}\bar{\mathbf{p}} - \bar{\mathbf{e}}, \tag{4.22}$$

where we have used eqn (3.33). By means of the expression:

$$\bar{\mathbf{e}} = -\frac{1}{\epsilon_0}(\bar{\mathbf{p}} \cdot \mathbf{n})\mathbf{n}, \tag{4.23}$$

which is obtained from eqns $(3.2)_2$ and $(3.34)_2$, eqn (4.22) reduces to

$$\dot{\bar{\mathbf{p}}} \times \mathbf{m} = -\frac{1}{\epsilon_0 \chi \gamma B_0}[\bar{\mathbf{p}} + \chi(\bar{\mathbf{p}} \cdot \mathbf{n})\mathbf{n}], \tag{4.24}$$

where $\mathbf{m}$ denotes the unit vector in the direction of $\mathbf{B}_0$. To solve eqn (4.24) we introduce the two unit vectors on the assumption that $\mathbf{B}_0$ is not parallel to $\mathbf{n}$:

$$\mathbf{k}_1 \equiv \frac{1}{\sin\theta}\mathbf{m} \times \mathbf{n}, \qquad \mathbf{k}_2 \equiv -\frac{1+\chi}{d}\cos\theta\,\mathbf{m} + \frac{1+\chi\cos^2\theta}{d}\mathbf{n} \tag{4.25}$$

where

$$d \equiv \sin\theta[(1 + 2\chi\cos^2\theta + \chi^2\cos^2\theta)]^{1/2}. \tag{4.26}$$

The two vectors are defined to be perpendicular to the vector $\mathbf{m} + \chi\cos\theta\mathbf{n}$ and to each other, so that $\bar{\mathbf{p}}$ being perpendicular to $\mathbf{m} + \chi\cos\theta\mathbf{n}$ can be expressed as

$$\bar{\mathbf{p}} = p_1\mathbf{k}_1 + p_2\mathbf{k}_2. \tag{4.27}$$

Substituting eqns (4.25) and (4.27) into eqn (4.24) and comparing the coefficients of $\mathbf{m}$, $\mathbf{n}$ and $\mathbf{m} \times \mathbf{n}$ in both members, we obtain

$$\frac{1+\chi\cos^2\theta}{d}\dot{p}_2 = \frac{1}{\epsilon_0\chi\gamma B_0\sin\theta}p_1, \tag{4.28}$$

$$\frac{1}{\sin\theta}\dot{p}_1 = -\frac{1+\chi}{\epsilon_0\chi\gamma B_0 d}p_2. \tag{4.29}$$

Under the condition $p_1 = p_0$, $p_2 = 0$ at $t = 0$, the solutions of the above equations are easily obtained, and $\bar{\mathbf{p}}$ is then expressed as

$$\bar{\mathbf{p}} = p_0\cos\omega_{EP}t\,\mathbf{k}_1 + rp_0\sin\omega_{EP}t\,\mathbf{k}_2, \tag{4.30}$$

where

$$\omega_{EP} \equiv \frac{1}{\epsilon_0\chi\gamma B_0}\left(\frac{1+\chi}{1+\chi\cos^2\theta}\right)^{1/2} \tag{4.31}$$

$$r \equiv \left[\frac{1+2\chi\cos^2\theta+\chi^2\cos^2\theta}{(1+\chi)(1+\chi\cos^2\theta)}\right]^{1/2} \tag{4.32}$$

and r is less than or equal to one. Thus $\bar{\mathbf{p}}$ is elliptically polarized on the plane being perpendicular to $\mathbf{m}+\chi\cos\theta\mathbf{n}$. The angular velocity of the rotation and the elliptic ratio are given by eqns (4.31) and (4.32), respectively, see Fig. 6. Figure 8 shows that the angular velocity takes the maximum value when $\mathbf{B}_0$ is perpendicular to $\mathbf{n}$, and that it becomes larger as χ becomes smaller. Figure 9 shows that when χ is large and $\mathbf{B}_0$ is perpendicular to $\mathbf{n}$, r is small, that is, $\bar{\mathbf{p}}$ is almost linearly polarized in the directions of $\pm\mathbf{k}_1$, and that when χ is small, r is nearly equal to one, that is, $\bar{\mathbf{p}}$ is almost circularly polarized.

When $\mathbf{B}_0$ is parallel to $\mathbf{n}$, $\bar{\mathbf{p}}$ is perpendicular to $\mathbf{n}$ in view of Fig. 7. Then the differential eqn (4.24) takes the same form as eqn (4.6). Thus by refering to eqn (4.8), $\bar{\mathbf{p}}$ is again expressed as eqn (4.30), where we define $\mathbf{k}_1$ as to be an arbitrary unit vector being perpendicular to $\mathbf{n}$ and $\mathbf{k}_2$ as to be $\mathbf{k}_1\times\mathbf{n}$ in this case. Here r is equal to one, that is, $\bar{\mathbf{p}}$ is precisely circularly polarized. The angular velocity takes the minimum value in this case from Fig. 8.

The variation of $\bar{\mathbf{e}}$ is given by substituting eqn (4.30) into eqn (4.23):

$$\bar{\mathbf{e}} = -\frac{rp_0\sin^2\theta}{\epsilon_0 d}\sin\omega_{EP}t\mathbf{n}. \tag{4.33}$$

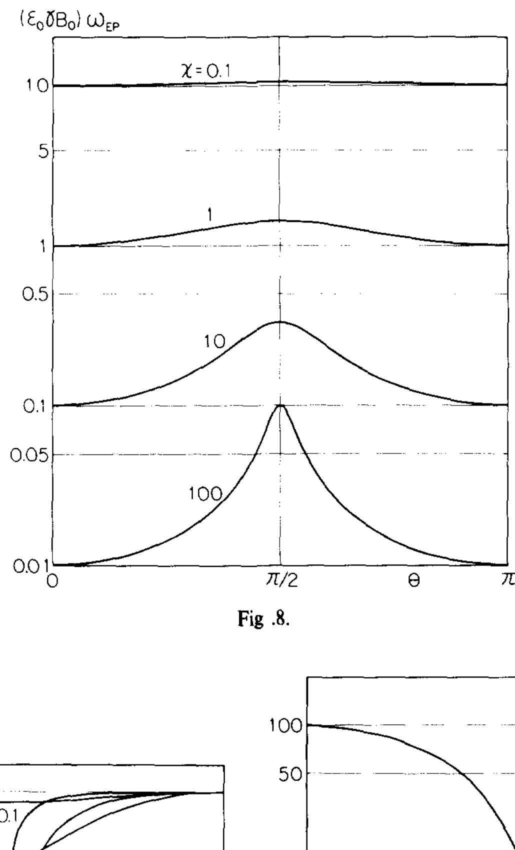

Fig .8.

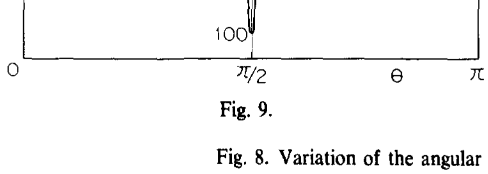

Fig. 9.

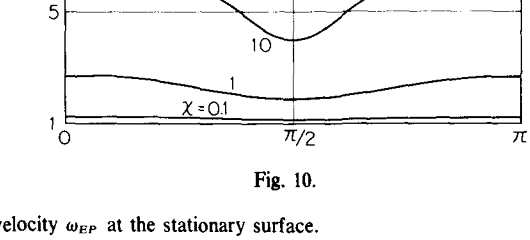

Fig. 10.

Fig. 8. Variation of the angular velocity ω_{EP} at the stationary surface.

Fig. 9. Variation of the elliptic ratio r at the stationary surface.

Fig. 10. Variation of the ratio of the maximum magnitude of $\bar{\mathbf{p}}$ to that of $\epsilon_0\bar{\mathbf{e}}$ at the stationary surface.

Thus $\bar{\mathbf{e}}$, being parallel to $\mathbf{n}$, periodically reverses in direction with the same period as the rotation of $\bar{\mathbf{p}}$. Figure 10 shows that the ratio of the maximum magnitude of $\bar{\mathbf{p}}$ to that of $\bar{\mathbf{e}}$ takes the maximum and the minimum values, respectively, when $\mathbf{B}_0$ is parallel and perpendicular to $\mathbf{n}$. We note that this situation is reversed in the case of the electro–magneto-polarization wave.

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